Int J Theor Phys (2010) 49: 2158–2163
DOI 10.1007/s10773-010-0401-7

Density Operator of Quantum Interference of Many Atoms in Spinor Bose-Einstein Condensates

Bao-Lai Zhang · Zhao-Xian Yu

Received: 10 April 2010 / Accepted: 1 June 2010 / Published online: 15 June 2010

Abstract In this letter, we have given the density operator of quantum interference of many atoms in spinor Bose-Einstein condensates.

Keywords Spinor Bose-Einstein condensates · Density operator

1 Introduction

Recent advance of experimental techniques on Bose-Einstein condensate (BEC) prompts us to closely and seriously look into theoretical possibilities which were mere imagination for theoreticians in this field. This is particularly true for spinor BEC where all hyperfine states of an atom Bose-condensed simultaneously, keeping these "spin" states degenerate and active. Recently, Barrett et al. [1] have succeeded in cooling Rb^{87} with the hyperfine state $F = 1$ by all optical methods without resorting to a usual magnetic trap in which the internal degrees of freedom is frozen. Since the spin interaction of the Rb^{87} atomic system is ferromagnetic, based on the refined calculation of the atomic interaction parameters by Klausen et al. [2], we now obtain concrete examples of the three-component spinor BEC $(F = 1, m_F = 1, 0, -1)$ for both antiferromagnetic (Na^{23}) [3] and ferromagnetic interaction cases. In the present spinor BEC the degenerate internal degrees of freedom play an essential role to determine the fundamental physical properties [4–13]. There is a rich variety of topological defect structures, which are already predicted in the earlier studies [14, 15] on the spinor BEC. Law et al. [16] constructed an excellent algebraic representation of the $F = 1$ BEC Hamiltonian to study the exact many-body states, and found that spin-exchange interactions cause a set of collective dynamic behavior of BEC. Recently, Zhang et al. [17] studied dynamic spin localization of spin-1 condensate driven by external magnetic fields.

B.-L. Zhang (✉)
Dongchang College, Liaocheng University, Liaocheng, Shandong Province 252000, China
e-mail: bao-laizhang@sohu.com

Z.-X. Yu
Department of Physics, Beijing Information Science and Technology University, Beijing 100192, China

Zou et al. [18] investigated quantum entanglement of many atoms in spinor Bose-Einstein condensates. In this letter, we shall study the density operator of quantum interference of many atoms in spinor Bose-Einstein condensates.

2 Model

We consider a dilute gas of trapped sodium atoms with hyperfine spin $f = 1$ trapped in a one-dimensional optical lattice, in which hyperfine states $|f = 1, m_f = 1\rangle$ and $|f = 1, m_f = -1\rangle$ are coupled by classical laser pulses. The Hamiltonian of this system is $H = H_0 + H_{int}$, where H_0 and H_{int} stand for the three-component condensate and the interaction between external fields and condensate, respectively:

$$
\begin{aligned}
H_0 = &\sum_{\alpha} \int d^3r \psi_\alpha^\dagger \left(\frac{-\nabla^2}{2M} + V(r) \right) \psi_\alpha + \frac{\lambda_s}{2} \sum_{\alpha,\beta} \int d^3r \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\alpha \psi_\beta \\
&+ \frac{\lambda_a}{2} \int d^3r (\psi_1^\dagger \psi_1^\dagger \psi_1 \psi_1 + \psi_{-1}^\dagger \psi_{-1}^\dagger \psi_{-1} \psi_{-1} + 2\psi_1^\dagger \psi_0^\dagger \psi_1 \psi_0 + 2\psi_{-1}^\dagger \psi_0^\dagger \psi_{-1} \psi_0 \\
&- 2\psi_1^\dagger \psi_{-1}^\dagger \psi_1 \psi_{-1} + 2\psi_0^\dagger \psi_0^\dagger \psi_1 \psi_{-1} + 2\psi_1^\dagger \psi_{-1}^\dagger \psi_0 \psi_0), \qquad (1)
\end{aligned}
$$

$$
H_{int} = \Omega \int d^3r (\psi_1^\dagger \psi_{-1} + \psi_{-1}^\dagger \psi_1), \qquad (2)
$$

where ψ_α is the atomic field annihilation operator associated with atom in the hyperfine spin state $|f = 1, m_f = \alpha\rangle (\alpha = -1, 0, 1)$, The mass of the atom is given by M, and $V(r) = V_0[\sin^2(kx) + \sin^2(ky) + \sin^2(kz)]$ is the optical lattice potential which is assumed to be the same for all three spin components. $k = 2\pi/\lambda$, λ is the wavelength of the laser field, and V_0 is the tunable depth of the potential well. A convenient measure for the strength V_0 of the lattice potential is the recoil energy $E_r = k^2/2M$. The coefficients λ_s and λ_a are related to s-wave scattering lengths a_0 and a_2 of two colliding bosons with total angular momenta 0 and 2 by $\lambda_s = 4\pi(a_0 + 2a_2)/3M$ and $\lambda_a = 4\pi(a_2 - a_0)/3M$. Ω is coupling strength between external fields and condensates.

In the Mott regime ($V_0 \gg E_r$), the potential barrier between all lattice sites is high so that no tunneling takes place through the optical lattice. the confining potential at a single lattice in this case can be well approximated by a harmonic potential $V(r) = V_0 k^2(x^2 + y^2 + z^2)$, and the condensate in the lattice can be confined to the ground state of harmonic potential. Therefore, the Hamiltonian of the system can be described by

$$
H_0 = (\lambda_s' + \lambda_a')N(N-1) - \lambda_a'(a_0^{\dagger 2} - 2a_1^\dagger a_{-1}^\dagger)(a_0^2 - 2a_1 a_{-1}), \qquad (3)
$$

$$
H_{int} = \Omega(a_1^\dagger a_{-1} + a_{-1}^\dagger a_1), \qquad (4)
$$

where $N = a_0^\dagger a_0 + a_1^\dagger a_1 + a_{-1}^\dagger a_{-1}$ is the total number of atoms in the condensate, which is a constant operator. $\lambda_i' = \lambda_i \int d^3r |\phi(r)|^4/2$ $(i = a, s)$, where $\phi(r)$ is the ground state of the trapped potential. Experiment [19] show that we can tune the Rabi frequency Ω within the large range by adjusting the laser parameters, i.e., $\Omega \gg N\lambda_a'$, where N is the number of atoms in the condensate. In the present paper, we shall neglect the term $N(N-1)$ of (3) because this term denotes a simple constant energy shift and has no contribution to the dynamics of the system.

In order to study how the system's dynamics is modified by the strong classical fields, one can introduce the dressed basis operators $b_1 = \frac{1}{\sqrt{2}}(a_1 + a_{-1})$, $b_{-1} = \frac{1}{\sqrt{2}}(a_1 - a_{-1})$, and switch to the interaction picture and consider the strong laser regime $\Omega \gg N\lambda'_a$ (so a rotating-wave approximation is used), then the Hamiltonian of the system can be expressed by [18]

$$H' = \Omega(b_1^\dagger b_1 - b_{-1}^\dagger b_{-1}) - \lambda'_a(a_0^{\dagger 2} a_0^2 + b_1^{\dagger 2} b_1^2 + b_{-1}^{\dagger 2} b_{-1}^2), \tag{5}$$

which is the starting Hamiltonian for the following discussion.

We start with the BEC prepared in the internal level $|f = 1, m_f = -1\rangle$, namely, the initial state of the system is

$$|\Phi(0)\rangle = |0\rangle_1 |0\rangle_0 |N\rangle_{-1}. \tag{6}$$

Based on (5), the state of the system at the time t becomes

$$|\Psi(t)\rangle = \exp(-iH't)|\Phi(0)\rangle = \sum_{m=-N/2}^{N/2} C_m e^{-i(2m\Omega - 2\lambda'_a m^2)t} \left|\frac{N}{2}, m\right\rangle_x |0\rangle_0, \tag{7}$$

where

$$\left|\frac{N}{2}, m\right\rangle_x = e^{-i\pi(a_{-1}^\dagger a_{-1} + a_1^\dagger a_1)/4} \left|\frac{N}{2} + m\right\rangle_1 \left|\frac{N}{2} - m\right\rangle_{-1}, \tag{8}$$

$$C_m = \sqrt{\frac{N!}{2^N(\frac{N}{2} + m)!(\frac{N}{2} - m)!}}. \tag{9}$$

3 Density Operator of Quantum Interference of Many Atoms in Spinor Bose-Einstein Condensates

The dissipation is included by considering the master equation

$$\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{H}', \hat{\rho}] + \sum_{j=0,\pm 1} \gamma_j (2\hat{b}_j \hat{\rho} \hat{b}_j^+ - \hat{b}_j^+ \hat{b}_j \hat{\rho} - \hat{\rho} \hat{b}_j^+ \hat{b}_j), \tag{10}$$

where γ_j $(j = 1, 2, 3)$ denotes the dissipation or loss rate due to some relaxation processes such as the coupling of the atoms in the system considered with the environment. For simplicity, in the following discussion, we will take $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. It is pointed out that the dissipation coming from the two-body process is not considered here for simplicity although it is also very important in the experimentally situation of interest. Note that the dissipation terms are identical to those commonly used in laser theory or measurement theory for continuous observation of interference fringes from the Bose-Einstein condensates but differ from those in the another atom detection scheme which amounts to neglecting the terms $2\hat{b}_j \hat{\rho} \hat{b}_j^\dagger$ in the above master equation. These terms may partially account for the loss due to particles escaping from the condensates, which can be understood by noting that the long-time asymptotical distribution of (10) is that with no particles at all in each condensate.

The master equation (10) can be solved exactly for any chosen initial state. In particular, we consider the case that the atoms in the system are initially in the coherent state (cs)

$\hat{\rho}(0) = |u_1, u_2)(u_1, u_2|$, and present the exact solution of the master (10) for such initial states. We expand the density operator at time t in the following form:

$$\rho(t) = \sum_{m_1,m_2}\sum_{n_1,n_2} \rho_{(m_1,m_2)}^{(n_1,n_2)}(t) e^{-i\pi(a_{-1}^\dagger a_{-1} + a_1^\dagger a_1)/4} \left|\frac{N}{2} + m_1\right\rangle_1 \left|\frac{N}{2} - m_2\right\rangle_{-1} |0\rangle_0$$
$$\times_0 \langle 0|_{-1}\left\langle \frac{N}{2} - n_2\right|_1 \left\langle \frac{N}{2} + n_1\right| e^{i\pi(a_{-1}^\dagger a_{-1} + a_1^\dagger a_1)/4} \tag{11}$$

so (11) can be rewritten by

$$\frac{\partial \rho_{(m_1,m_2)}^{(n_1,n_2)}(t)}{\partial t} = (-i\alpha_1 + \alpha_2)\rho_{(m_1,m_2)}^{(n_1,n_2)}(t) + \alpha_3 \rho_{(m_1-1,m_2-1)}^{(n_1,n_2)}(t)$$
$$+ \alpha_4 \rho_{(m_1+1,m_2+1)}^{(n_1,n_2)}(t)$$
$$+ \alpha_5 \rho_{(m_1-2,m_2-2)}^{(n_1,n_2)}(t) + \alpha_6 \rho_{(m_1+2,m_2+2)}^{(n_1,n_2)}(t)$$
$$+ \alpha_7 \rho_{(m_1,m_2)}^{(n_1-1,n_2-1)}(t) + \alpha_8 \rho_{(m_1,m_2)}^{(n_1+1,n_2+1)}(t)$$
$$+ \alpha_9 \rho_{(m_1,m_2)}^{(n_1-2,n_2-2)}(t) + \alpha_{10} \rho_{(m_1,m_2)}^{(n_1+2,n_2+2)}(t)$$
$$+ \alpha_{11} \rho_{(m_1+1,m_2)}^{(n_1+1,n_2)}(t) + \alpha_{12} \rho_{(m_1,m_2-1)}^{(n_1,n_2-1)}(t), \tag{12}$$

where

$$\alpha_1 = \frac{\lambda_a' \Omega}{2}[(n_1 - m_1)(N + m_1 + n_1 - 1) - (n_2 - m_2)(3N - m_2 - n_2 - 1)$$
$$+ (N + 2n_1)(N - 2n_2) - (N + 2m_1)(N - 2m_2)], \tag{13}$$

$$\alpha_2 = -\gamma(2N + m_1 + n_1 - m_2 - n_2)t, \tag{14}$$

$$\alpha_3 = \Omega\sqrt{\left(\frac{N}{2} + m_1\right)\left(\frac{N}{2} - m_2 + 1\right)}, \tag{15}$$

$$\alpha_4 = \Omega\sqrt{\left(\frac{N}{2} + m_1 + 1\right)\left(\frac{N}{2} - m_2\right)}, \tag{16}$$

$$\alpha_5 = -\Omega\sqrt{\left(\frac{N}{2} + m_1\right)\left(\frac{N}{2} + m_1 - 1\right)\left(\frac{N}{2} - m_2 + 1\right)\left(\frac{N}{2} - m_2 + 2\right)}, \tag{17}$$

$$\alpha_6 = -\Omega\sqrt{\left(\frac{N}{2} + m_1 + 2\right)\left(\frac{N}{2} + m_1 + 1\right)\left(\frac{N}{2} - m_2 - 1\right)\left(\frac{N}{2} - m_2\right)}, \tag{18}$$

$$\alpha_7 = -\Omega\sqrt{\left(\frac{N}{2} + n_1\right)\left(\frac{N}{2} - n_2 + 1\right)}, \tag{19}$$

$$\alpha_8 = -\Omega\sqrt{\left(\frac{N}{2} + n_1 + 1\right)\left(\frac{N}{2} - n_2\right)}, \tag{20}$$

$$\alpha_9 = \Omega\sqrt{\left(\frac{N}{2}+n_1\right)\left(\frac{N}{2}+n_1-1\right)\left(\frac{N}{2}-n_2-1\right)\left(\frac{N}{2}-n_2\right)}, \tag{21}$$

$$\alpha_{10} = -\Omega\sqrt{\left(\frac{N}{2}+n_1+2\right)\left(\frac{N}{2}+n_1+1\right)\left(\frac{N}{2}-n_2-1\right)\left(\frac{N}{2}-n_2\right)}, \tag{22}$$

$$\alpha_{11} = 2\gamma\sqrt{\left(\frac{N}{2}+m_1+1\right)\left(\frac{N}{2}+n_1+1\right)}, \tag{23}$$

$$\alpha_{12} = 2\gamma\sqrt{\left(\frac{N}{2}-m_2+1\right)\left(\frac{N}{2}-n_2+1\right)}. \tag{24}$$

The solution of (12) is

$$\rho^{(n_1,n_2)}_{(m_1,m_2)}(t) = \frac{u_1^{\frac{N}{2}+m_1} u_1^{*\frac{N}{2}+n_1} u_2^{\frac{N}{2}-m_2} u_2^{*\frac{N}{2}-n_2}}{\sqrt{(\frac{N}{2}+m_1)!(\frac{N}{2}-m_2)!(\frac{N}{2}+n_1)!(\frac{N}{2}-n_2)!}} e^{\eta+\xi}, \tag{25}$$

where $\eta = -i\alpha_1 + \alpha_2$, $\xi = \sum_{i=1}^{10} \xi_i$, and ξ_i are respectively:

$$\xi_1 = \Omega\left|\frac{u_2}{u_1}\right|\left[\left(\frac{N}{2}+m_1\right)t\right]e^{i(\varphi_2-\varphi_1)}, \tag{26}$$

$$\xi_2 = \Omega\left|\frac{u_1}{u_2}\right|\left[\left(\frac{N}{2}-m_2\right)t\right]e^{i(\varphi_1-\varphi_2)}, \tag{27}$$

$$\xi_3 = -\Omega\left|\frac{u_2}{u_1}\right|^2\left[\left(\frac{N}{2}+m_1\right)\left(\frac{N}{2}+m_1-1\right)t\right]e^{2i(\varphi_2-\varphi_1)}, \tag{28}$$

$$\xi_4 = -\Omega\left|\frac{u_1}{u_2}\right|^2\left[\left(\frac{N}{2}-m_2\right)\left(\frac{N}{2}-m_2-1\right)t\right]e^{2i(\varphi_1-\varphi_2)}, \tag{29}$$

$$\xi_5 = -\Omega\left|\frac{u_2}{u_1}\right|\left[\left(\frac{N}{2}+n_1\right)t\right]e^{i(\varphi_1-\varphi_2)}, \tag{30}$$

$$\xi_6 = -\Omega\left|\frac{u_1}{u_2}\right|\left[\left(\frac{N}{2}-n_2\right)t\right]e^{i(\varphi_2-\varphi_1)}, \tag{31}$$

$$\xi_7 = \Omega\left|\frac{u_2}{u_1}\right|^2\left[\left(\frac{N}{2}+n_1\right)\left(\frac{N}{2}+n_1-1\right)t\right]e^{2i(\varphi_1-\varphi_2)}, \tag{32}$$

$$\xi_8 = \Omega\left|\frac{u_1}{u_2}\right|^2\left[\left(\frac{N}{2}-n_2\right)\left(\frac{N}{2}-n_2-1\right)t\right]e^{2i(\varphi_2-\varphi_1)}, \tag{33}$$

$$\xi_9 = 2\gamma t|u_1|^2, \qquad \xi_{10} = 2\gamma t|u_2|^2, \tag{34}$$

where we have used $u_1 = |u_1|e^{i\varphi_1}$ and $u_2 = |u_2|e^{i\varphi_2}$.

References

1. Barrett, M., et al.: Phys. Rev. Lett. **87**, 010404 (2001)

2. Klausen, N.N., et al.: Phys. Rev. A **64**, 053602 (2001)
3. Stenger, J., et al.: Nature **369**, 345 (1998)
4. Yi, S., Mustecaplioglu, O.E., Sun, C.P., You, L.: Phys. Rev. A **66**, 011601(R) (2002)
5. Mustecaplioglu, O.E., Zhang, M., Yi, S., You, L., Sun, C.P.: Phys. Rev. A **68**, 063616 (2003)
6. Xie, Z.W., Zhang, W.P., Chui, S.T., Liu, W.M.: Phys. Rev. A **69**, 053609 (2004)
7. Wu, Y., Yang, X., Sun, C.P.: Phys. Rev. A **62**, 063603 (2000)
8. Wu, Y.: Phys. Rev. A **54**, 4534 (1996)
9. Wu, Y., Yang, X., Xiao, Y.: Phys. Rev. Lett. **86**, 2200 (2001)
10. Liu, W.M., Fan, W.B., Zheng, W.M., Liang, J.Q., Chui, S.T.: Phys. Rev. Lett. **88**, 170408 (2002)
11. Liu, W.M., Wu, B., Niu, Q.: Phys. Rev. Lett. **84**, 2294 (2000)
12. Niu, Q., Wang, X.D., Kleinman, L., Liu, W.M., Nicholson, D.M.C., Stocks, G.M.: Phys. Rev. Lett. **83**, 207 (1999)
13. Liang, J.J., Liang, J.Q., Liu, W.M.: Phys. Rev. A **68**, 043605 (2003)
14. Ohmi, T., Machida, K.: J. Phys. Soc. Jpn. **67**, 1822 (1998)
15. Ho, T.L.: Phys. Rev. Lett. **81**, 742 (1998)
16. Law, C.K., Pu, H., Bigelow, N.P.: Phys. Rev. Lett. **81**, 5257 (1998)
17. Zhang, P., Zhang, A.Z., Duan, S.Q., Zhao, X.G.: Phys. Lett. A **286**, 107 (2001)
18. Zou, X.B., et al.: Phys. Rev. A **69**, 053608 (2004)
19. Mandel, O., et al.: Phys. Rev. Lett. **91**, 010407 (2003)